J. Fluid Mech. (1978), vol. 87, part 4, pp. 737–748
Printed in Great Britain

Some predictions from the mean-field thermosolutal equations

By N. RIAHI

Department of Mathematics, Winthrop College, Rock Hill, South Carolina 29733

(Received 29 January 1976 and in revised form 26 November 1977)

Nonlinear thermosolutal convection is investigated using the mean-field approximation (Herring 1963; Busse 1970). The boundary-layer method for rigid boundaries is used by assuming a large Rayleigh number R for different ranges of the diffusivity ratio τ and the solute Rayleigh number R_S. The heat and solute fluxes F and F_S are determined for the values of the wavenumbers α_n which optimize F. The α_N mode (where N denotes the total number of modes) is shown to have a solute-layer thickness of order $\tau^{\frac{1}{3}}\delta_{tN}$ (δ_{tN} denoting the temperature-layer thickness in the α_N mode), and it is also proved that it is only for this mode that the solute concentration affects the boundary-layer structure. Solutions are possible if $K = F_S R_S/FR < 1$. For $K \ll 1$, the stabilizing effect of solute is unimportant and there can be infinitely many modes. However, as $K \to 1-$, N, α_n, F and F_S decrease rapidly and the maximizing convection is suppressed entirely by the solute concentration. A simple interpretation of the model for the diffusive system leads also to the results for the salt-finger system.

1. Introduction

Double-diffusive convection is the convective motion of fluids in which there are gradients of two properties. The motion depends strongly for its driving mechanism on the different diffusive properties associated with the stabilizing and destabilizing forces. For the case in which heat and any solute are the properties, the problem has been named thermosolutal convection.

Double-diffusive convection is important in many areas of geophysics, astrophysics and engineering, and has been observed in nature. Two examples in oceanography are (i) layers where colder fresher water overlies warm salty water, which are found underneath drifting ice islands in the Arctic Ocean (Neal, Neshyba & Denner 1969; Neshyba, Neal & Denner 1971), and (ii) layers of hot salty water bounded by diffusive interfaces, which are found near the bottom in parts of the Red Sea of various depths (Degens & Ross 1969). These are nearly saturated with salts of geothermal origin, including a high proportion of heavy metals which are of commercial value. An example in astrophysics is the helium-rich core of some stars, in which the fluid is heated from below and is transported upwards by double-diffusive convection. For a more detailed discussion of double-diffusive phenomena and their applications, the reader is referred to Turner (1973, 1974).

This paper studies nonlinear double-diffusive convection under the so-called mean-field approximation of the equations for momentum, heat and solute. Briefly, these equations are derived by ignoring the interaction between the fluctuation quantities but retaining the interaction between the mean and the fluctuation quantities. For a more detailed discussion of these equations and their derivation, we refer to the papers by Herring (1963) and Busse (1970). Previous studies of these equations for the case of thermal convection have shown that, for moderate or large values of the Prandtl number Pr, as far as the statistical properties of motion are concerned the results derived do not differ appreciably from the experimental results based on the original equations.

On the basis of the postulate first proposed by Malkus (1954), we assume that the maximized heat transport F is the one realized in the diffusive regime (defined as the one in which the energy driving the flow comes from the component having the larger diffusivity). For the salt-finger regime (the opposite case), the relevant postulate is that the flow fields tend to maximize the solute transport F_S. The success of the previous studies of thermal convection based on Malkus' postulate encouraged us to modify this postulate for our problem. A discussion of this can be found in the papers by Lindberg (1971) and Straus (1972). In the latter paper it is found, for example, that in the salt-finger case the mode which maximizes F_S lies within the wave band of the stable modes; this suggests a closed relation of the stability of a particular flow to its ability to transport salt across the layer.

The treatment here is for the steady case. Numerical studies by Veronis (1965, 1968) and Straus (1972) of the diffusive and salt-finger regimes of thermosolutal convection indicate that a steady state can be reached by a convective flow of finite amplitude (away from its linear instability regime). Of course, sufficiently strong convective flows are time dependent, but the present study aims at exploring the properties of nonlinear thermosolutal convection in the simpler case of a steady state, which may also be considered as an approximation in some sense.

Since the studies on upper bounds in double-diffusive convection by Lindberg (1971) and Straus (1974) are based on the so-called power integrals, which may be derived from the mean-field equations alone, we may conclude, as Chan did for thermal convection (Chan 1971, henceforth referred to as I), that the optimized quantities of the present study (which uses the mean-field equations themselves) are one step closer to their true values, at least for $Pr \gg 1$.

Our study is also the first attempt to apply multi-boundary-layer techniques to double-diffusive convection to determine the optimal flow quantities of the maximized fields for sufficiently large values of the Rayleigh number R in the diffusive system or of the solute Rayleigh number R_S in the salt-finger system. This technique was first formulated by Busse (1969). In improving the upper bound on the heat flux, Busse considered a sequence of different boundary layers by adjusting the horizontal scale from its interior value to its boundary value. The thickness of each boundary layer was assumed to be large in comparison with the thickness of the next layer closer to the surface, and the convecting component of the heat flux was assumed to be approximately equal to the total heat flux (the conducting component was small) in all but the last of the boundary layers, where it was smaller, but still of the same order as the total heat flux. Later on, Chan (1971) used Busse's technique to study turbulent convection at infinite Prandtl number and obtained the preferred upper bound on the heat transport. Since then, this technique has been used by Busse & Joseph (1972), Gupta & Joseph (1973), Chan (1974) and Riahi (1977) to study nonlinear convection. In all such studies, a schematic structure for all the modes was considered. Also, it was assumed that higher modes have shorter length scales and that coupling among the different modes occurs only between the $(n+1)$th and the nth mode in the nth boundary layer. The multi-modal regime and details of the solutions of our governing equations (§ 2) are given in § 3.

Unless otherwise stated, the basic model treated here is for the diffusive regime. In § 4, a simple modification is made so that the results can be applied to the salt-finger regime as well.

2. Governing equations

We consider an infinite horizontal layer of fluid of depth d bounded above and below by two rigid, perfectly conducting planes maintained at temperatures T_0 and $T_0 + \Delta T$ ($\Delta T > 0$) and at solute concentrations S_0 and $S_0 + \Delta S$ ($\Delta S > 0$), respectively. The mean-field equations are derived from the Boussinesq equations for momentum, heat and solute when all nonlinear terms are neglected with the exception of those which enter the equations for the horizontally averaged temperature and solute concentration (Busse 1970). The non-dimensional steady-state forms of these equations, after eliminating the pressure and horizontal velocity components, are

$$\nabla^4 W + R\nabla_1^2 T - R_S \nabla_1^2 S = 0, \tag{1}$$

$$\nabla^2 T + (1 - \overline{WT} + \langle WT \rangle)\, W = 0, \tag{2}$$

$$\tau^2 \nabla^2 S + (\tau - \overline{WS} + \langle WS \rangle)\, W = 0. \tag{3}$$

Here S is the deviation of the solute concentration from its horizontal average, W is the vertical component of the velocity vector, T is the deviation of the temperature from its horizontal average, the bars denote horizontal averages, the angle brackets denote a further vertical average over the whole layer, and $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Also, $R = \alpha g d^3 \Delta T / K_t \nu$ is the Rayleigh number, $R_S = \beta g d^3 \Delta S / K_t \nu$ is the so-called solute Rayleigh number and $\tau = K_S/K_t$ is the ratio of the diffusivity coefficients K_S and K_t, of solute and heat respectively, where α is the coefficient of thermal expansion, β is the fractional change in density due to a change in the solute concentration, ν is the kinematic viscosity and g is the acceleration due to gravity.

We shall rescale our dependent variables such that

$$\omega = (FR)^{-\frac{1}{2}}\, W, \quad \theta = (R/F)^{\frac{1}{2}}\, T, \quad C = (FR)^{\frac{1}{2}} F_S^{-1} S, \tag{4}$$

where $F = \langle WT \rangle$ and $F_S = \langle WS \rangle$ are the heat and solute fluxes, respectively. The governing differential equations are now

$$\nabla^4 \omega + \nabla_1^2 \theta - K \nabla_1^2 C = 0, \tag{5}$$

$$\frac{1}{FR} \nabla^2 \theta + \left(1 - \overline{\omega\theta} + \frac{1}{F}\right) \omega = 0, \tag{6}$$

$$\frac{\tau^2}{FR} \nabla^2 C + \left(1 - \overline{\omega C} + \frac{\tau}{F_S}\right) \omega = 0, \tag{7}$$

where $$K = F_S R_S / FR. \tag{8}$$

We shall use the following constraints to determine F and F_S:

$$F = \frac{1 - R^{-1}\langle|\nabla\theta|^2\rangle}{\langle(1-\overline{\omega\theta})^2\rangle}, \quad F_S = \frac{1-(\tau F_S/FR)\langle|\nabla C|^2\rangle}{\langle(1-\overline{\omega C})^2\rangle}, \tag{9}$$

which are derived by multiplying (6) and (7) by θ and C respectively and taking the total average over the whole layer.

The boundary conditions appropriate to rigid surfaces at $z = 0, 1$ are

$$\omega = \partial\omega/\partial z = \theta = C = 0. \tag{10}$$

The usual form of the cellular structure for the dependent variables is assumed, i.e.

$$(\omega, \theta, C) = \sum_n [\omega_n(z), \theta_n(z), C_n(z)]\, \phi_n(x, y), \tag{11}$$

where ϕ_n can be any solution of

$$\nabla_1^2\phi_n(x,y) = -\alpha_n^2\phi_n(x,y),$$

for some horizontal wavenumber α_n. Functions with different wavenumbers are naturally orthogonal, and here are chosen to be orthonormal. This separation of variables leads us to the system of nonlinear ordinary differential equations

$$\left(\frac{d^2}{\partial z^2} - \alpha_n^2\right)^2 \omega_n - \alpha_n^2(\theta_n - KC_n) = 0, \tag{12}$$

$$\frac{1}{FR}\left(\frac{d^2}{\partial z^2} - \alpha_n^2\right)\theta_n + \left(1 - \sum_n \omega_n\theta_n + \frac{1}{F}\right)\omega_n = 0, \tag{13}$$

$$\frac{\tau^2}{FR}\left(\frac{d^2}{dz^2} - \alpha_n^2\right)C_n + \left(1 - \sum_n \omega_n C_n + \frac{\tau}{F_S}\right)\omega_n = 0, \tag{14}$$

with boundary conditions

$$\omega_n = d\omega_n/dz = \theta_n = C_n = 0 \quad \text{at} \quad z = 0, 1. \tag{15}$$

Equations (12)–(15) must then be solved subject to (9). We shall obtain the solutions by using the multi-boundary-layer method, treating R as a large parameter.

3. Multi-modal regime

We refer to I for details on the mathematical analysis of the multi-modal solutions.

The case $\tau \ll 1$

Each α_n mode ($n = 1, \ldots, N-1$) has three regions: the interior, the intermediate layer and the inner layer. The α_N mode has four regions: the interior, the intermediate layer, the thermal layer and the solute layer. The interior of each mode coincides with the inner layer of the previous mode. Coupling among the different modes occurs only between the nth and the $(n-1)$th mode in the $(n-1)$th boundary layer. It is assumed that

$$\delta_{SN} \ll \delta_{tN} \ll 1/\alpha_N \ll \delta_{N-1} \ll \ldots \ll \delta_n \ll 1/\alpha_n \ll \delta_{n-1} \ll \ldots \ll \delta_1 \ll 1/\alpha_1 \quad \text{as} \quad \alpha_n \to \infty, \tag{16}$$

where $1/\alpha_n$ and δ_n are the thicknesses of the intermediate and inner layers, respectively, and δ_{SN} and δ_{tN} are the thicknesses of the solute and thermal layers of the α_N mode, respectively. The boundary-layer structures of all but the last mode are found to be essentially the same as those for thermal convection alone (I).

In the interior of the α_n mode ($n = 1, \ldots, N$), we define $\zeta_{n-1} = z/\delta_{n-1}$ as the boundary-layer variable for the $(n-1)$th mode. Equation (12) then gives

$$\alpha_n^2 \omega_n = \theta_n - KC_n. \tag{17}$$

In the intermediate layer of the α_n mode ($n = 1, \ldots, N$), we define $\xi_n = \alpha_n z$ as the variable. Since there is no coupling between the modes in this layer and conductive terms are not yet important, (13) and (14) give

$$\omega_n \theta_n = 1, \quad \omega_n C_n = 1. \tag{18}$$

We then find from (12) and (18) that

$$\omega_n = \frac{(1-K)^{\frac{1}{2}}}{\alpha_n} \xi_n^2 \left(\log \frac{1}{\xi_n}\right)^{\frac{1}{2}} \quad \text{as} \quad \xi_n \to 0. \tag{19}$$

In the inner layer of the α_n mode ($n = 1, \ldots, N-1$), we define $\zeta_n = z/\delta_n$ as the variable. We then find from the governing equations (12)–(14), after applying matching conditions (matching the solutions to the corresponding solutions in the intermediate layer) and a similar procedure to that in I, that

$$\omega_n = A_n \zeta_n^2, \tag{20}$$

$$A_n \theta_n = A_n C_n = \begin{cases} B_n^{\frac{1}{3}}(3^{-\frac{1}{3}} B_n^{\frac{1}{3}} \zeta_n - \frac{1}{12} B_n^{\frac{2}{3}} \zeta_n^4) & \text{for} \quad 0 \leqslant \zeta_n \leqslant (12/B_n)^{\frac{1}{6}}, \\ 1/\zeta_n^2 \quad \text{for} \quad (12/B_n)^{\frac{1}{6}} \leqslant \zeta_n, \end{cases} \tag{21}$$

where

$$A_n = (1-K)^{\frac{1}{2}} \alpha_n \delta_n^2 [\log(1/\alpha_n \delta_n)]^{\frac{1}{2}}, \tag{22}$$

$$B_n = \alpha_{n+1}^4 \alpha_n^2 \delta_n^6 \log(1/\alpha_n \delta_n). \tag{23}$$

Thus the solutions for the vertical velocity and temperature in the α_n modes ($n = 1, \ldots, N-1$) are very similar to the corresponding ones in thermal convection (I), and the solution for the solute concentration has essentially the same form as that for the temperature in these modes. In the temperature layer of the α_N mode, we define $\eta_N = z/\delta_{tN}$ as the layer variable. Applying matching conditions, we find from (12)–(14) that

$$\omega_N = A_n \eta_N^2, \tag{24}$$

$$C_N = 1/A_N \eta_N^2, \tag{25}$$

$$\theta_N = \frac{P_N \eta_N}{3A_N} \int_0^1 (1-t^2)^{-\frac{1}{3}} \exp(-\tfrac{1}{3} P_N \eta_N^3 t)\, dt, \tag{26}$$

where

$$P_N^2 = (1-K) FR\, \delta_{tN}^6 \alpha_N^2 \log(1/\alpha_N \delta_{tN}), \quad A_N = (1-K)^{\frac{1}{2}} \alpha_N \delta_{tN}^2 [\log(1/\alpha_N \delta_{tN})]^{\frac{1}{2}}. \tag{27}$$

In the solute layer of the α_N mode, we define the layer variable as $\zeta_N = Z/\delta_{SN}$. The governing equations and matching conditions then give

$$\omega_N = E_N \zeta_N^2, \tag{28}$$

$$C_N = \frac{P_N \zeta_N}{3E_N} \int_0^1 (1-t^2)^{-\frac{1}{3}} \exp\left(-\tfrac{1}{3}\zeta_N^3 P_N t\right) dt, \tag{29}$$

$$\theta_N = (h_N/E_N)\,\eta_N, \tag{30}$$

where
$$E_N = (\delta_{SN}/\delta_{tN})^2 A_N, \tag{31}$$

P_N is given in (27) and h_N is a constant of order one whose numerical value is not needed here. Thus in the α_N mode the solutions for velocity and temperature are very similar to the corresponding ones in thermal convection in the first three regions and are essentially unchanged in the δ_{SN} layer. The solute concentration has essentially the same form as the temperature in the interior and $1/\alpha_N$ layer, is essentially unchanged in the δ_{tN} layer and has a similar form to the temperature in the ζ_{SN} layer.

To determine F and F_S, we evaluate the expressions $\langle|\nabla\theta|^2\rangle$, $\langle|\nabla C|^2\rangle$, $\langle(1-\overline{\omega\theta})^2\rangle$ and $\langle(1-\overline{\omega C})^2\rangle$ in (9). After use of a formal procedure to maximize F (I, § 5), we then find that

$$\alpha_N = b_N[(1-K)\,R]^{\frac{1}{6}(2-5/10^n)} \prod_{I=1}^{n-1} \left(\log\frac{1}{g_I}\right)^{\frac{1}{2}(10^{I-n})}, \tag{32}$$

$$g_N^{2/(10^n)} \left(\log\frac{1}{g_n}\right)^{4(10^{n-1})} \prod_{I=1}^{n-1} \left(\log\frac{1}{g_I}\right)^{-6(10^{I-1})} (1-K)\,R = 1, \tag{33}$$

$$F = K_N[(1-K)\,R]^{\frac{1}{3}(1-10^{-N})} \prod_{I=1}^{N} \left(\log\frac{1}{g_I}\right)^{\frac{1}{5}(10^{I-N})}, \tag{34}$$

$$\delta_{SN} = \tau^{\frac{1}{3}}\delta_{tN}, \tag{35}$$

$$F_S = F/[\sigma(\tau^{-\frac{2}{3}}-\tau)\,(K_N^{\frac{5}{2}}/b_N)^{\frac{1}{3}}+\tau], \tag{36}$$

where
$$\left.\begin{aligned} &\prod_{I=1}^{0} = 1, \quad \delta_n = g_n/\alpha_n, \quad \delta_{tN} = g_N/\alpha_N, \quad \sigma = 2{\cdot}2212, \\ &b_{n+1} = 10^{\frac{1}{27}(9n-10+10^{-n+1})}(2/\beta)^{3(10^{-n})}\, b_1^{13(10^{-n})}\,(2b_1^4/\beta)^{\frac{1}{3}(1-10^{-n+1})}, \quad 0 \leqslant n \leqslant N-1, \end{aligned}\right\} \tag{37}$$

$$K_N = \sigma^{-\frac{6}{5}}(1+\tfrac{1}{3}b_1^4-\tfrac{4}{3}(10^{N-1})\,b_1^4)^{\frac{6}{5}}\, b_N^{\frac{2}{5}}, \tag{38}$$

$\beta = 0{\cdot}4539$, and b_1^4 is the root of the equation

$$G \equiv \left\{1 - \frac{R_S\tau^{\frac{2}{3}}}{R[(1-\tau^{\frac{5}{3}})\,(1+\frac{1}{3}b_1^4-\frac{4}{3}(10^{N-1})\,b_1^4)+\tau^{\frac{5}{3}}]}\right\} \left\{\frac{1+\frac{1}{3}b_1^4-\frac{4}{3}(10^N)\,b_1^4}{1+\frac{1}{3}b_1^4-\frac{4}{3}(10^{N-1})\,b_1^4}\right\}$$
$$-\tfrac{1}{3}(10^N-1)\,R_S\tau^{\frac{2}{3}}\,b_1^4(1-\tau^{\frac{5}{3}})\,R^{-1}\,[(1-\tau^{\frac{5}{3}})\,(1+\tfrac{1}{3}\,b_1^4-\tfrac{4}{3}\,(10^{N-1})\,b_1^4)+\tau^{\frac{5}{3}}]^{-2} = 0. \tag{39}$$

From (8), (34) and (38) we obtain the condition for the validity of the solutions as

$$R_S\tau^{\frac{2}{3}}R^{-1}[(1-\tau^{\frac{5}{3}})\,(1+\tfrac{1}{3}b_1^4-\tfrac{4}{3}(10^{N-1})\,b_1^4)+\tau^{\frac{5}{3}}]^{-1} < 1. \tag{40}$$

It turns out that (32)–(40) are valid for all possible ranges of τ. Equations (8), (36), (39) and (40) could be further simplified using the condition $\tau \ll 1$. If we simplify (39) using $\tau \ll 1$, it becomes a quadratic equation for b_1^4 with two real and positive roots. To maximize F, we use the root which gives the relative maximum of F. It is easily seen that the results are consistent with the thermal convection problem in I for $K \ll 1$. For $K \sim 1$ and as $K \to 1$ in this range, the number of modes, as well as F and F_S, decreases rapidly. Using (16) and (32)–(33), we have

$$N = (\log 10)^{-1}\{\log(\tfrac{3}{2}\log((1-K)\,R))\}. \tag{41}$$

As $1-K \to 1/R$, N becomes finite and approaches zero. Thus for sufficiently small $|1-K-1/R|$, nonlinear maximizing convection is inhibited entirely by the solute concentration.

The case $\tau = O(1)$

It is found that for this case the δ_{SN} layer merges with the δ_{tN} layer. Equations (17)–(24), (26)–(27) and (32)–(41) are valid here. However, the solution for the solute concentration in the temperature layer of the α_N mode takes the form

$$C_N = \frac{P_N \eta_N}{3\tau A_N} \int_0^1 (1-t^2)^{-\frac{2}{3}} \exp\left(-P_N \eta_N^3 t/3\tau\right) dt. \tag{42}$$

In particular, for $\tau = 1$ we find from (39) that $b_1^4 = 3/(4 \times 10^N - 1)$ as expected since the problem could be reparameterized to be equivalent to a singly diffusive case in which $b_1^4 = 3/(4 \times 10^N - 1)$ (cf. I). For $\tau > 1$, $G \geqslant 0$ if $b_1^4 = 3/(4 \times 10^N - 1)$ and $G < 0$ if $b_1^4 = 3/(4 \times 10^{N-1} - 1)$. Hence there is always one positive root between these two values of b_1^4. Similarly, for $\tau < 1$, there is always one positive root for b_1^4 in the interval $[0, 3/(4 \times 10^N - 1)]$. When there is more than one valid positive root, we choose the one which gives the relative maximum of F.

The case $\tau \gg 1$

If $\tau \gg 1$, then $\delta_{SN} \gg \delta_{tN}$. Equations (17)–(23) and (32)–(40) are valid here. However, we have the following solutions in the solute layer after using (12)–(14) and the matching conditions:

$$\omega_N = E_N \zeta_N^2, \quad \theta_N = E_N^{-1} \zeta_N^{-2}, \tag{43), (44}$$

$$C_N = \frac{Q_N \zeta_N}{3E_N} \int_0^1 (1-t^2)^{-\frac{2}{3}} \exp\left(-\tfrac{1}{3} Q_N \zeta_N^3 t\right) dt, \tag{45}$$

where

$$Q_N^2 = (1-K)\, FR\, \delta_{tN}^6 \alpha_N^2 \left(\log \frac{1}{\alpha_N \delta_{SN}}\right), \quad E_N = (1-K)^{\frac{1}{2}} \alpha_N \delta_{SN}^2 \left(\log \frac{1}{\alpha_N \delta_{SN}}\right)^{\frac{1}{2}}. \tag{46}$$

Similarly, we obtain the following solutions in the temperature layer:

$$C_N = h_N \zeta_N / A_N, \tag{47}$$

$$\theta_N = \frac{Q_N \eta_N}{3A_N} \int_0^1 (1-t^2)^{-\frac{2}{3}} \exp\left(-\tfrac{1}{3} Q_N \eta_N^3 t\right) dt, \tag{48}$$

where ω_N and A_N are given by (24) and (27), respectively, and Q_N is given by (46). The boundary-layer structure is now valid if

$$\tau \ll [(1-K)\, R]^{\frac{2}{3} \times 10^{-N}}. \tag{49}$$

Otherwise, the δ_{SN} layer merges with the $1/\alpha_N$ layer and the boundary-layer structure breaks down. Using the condition

$$\delta_{tN} \ll \delta_{SN} \ll 1/\alpha_N \ll \delta_{N-1} \quad \text{as} \quad R \to \infty,$$

we obtain the value of N which maximizes F as

$$N = \frac{1}{\log 10} \left\{ \log \left[\frac{\log [(1-K)\, R]^{\frac{2}{3}}}{\log \tau} \right] \right\}. \tag{50}$$

For $K \ll 1$, there are infinitely many modes, but for $K \sim 1$ and as $1-K \to \tau^{\frac{2}{3}}/R$, N becomes finite and approaches zero. Thus for sufficiently small values of $|1-K-\tau^{\frac{2}{3}}/R|$, nonlinear maximizing convection is inhibited by the stabilizing effect of solute.

4. Discussion

The boundary-layer analysis has shown that, for given R and R_S, the fluxes F and F_S are continuous functions of τ. In general, solute and temperature have identical inner layers in the α_n mode ($n = 1, \ldots, N-1$), but for $\tau \gg 1$ or $\tau \ll 1$, solute and temperature have different inner layers in the α_N mode. It is found from (32)–(34) that the relation $F \sim 1/\delta_{tN}$ holds for the strongly convective case ($K \ll 1$), as in thermal convection problems at high R. However, for $K \sim 1$, the relation between F and δ_{tN} depends also on K. Detailed calculations of F and F_S indicate that δ_{tN} has essentially the unique role of fixing and determining F. By analogy, δ_{SN} should have the role of fixing and determining F_S in the salt-finger regime. It is noted from (32), (33) and (35) that δ_n ($n = 1, \ldots, N-1$) and δ_{tN} depend on R and K and that τ and R_S are not free parameters. However, δ_{SN} depends strongly on τ as well as on R and K. If $\tau \sim 1$, $\delta_{SN} \sim \delta_{tN}$ and either $\tau \gg 1$ or $\tau \ll 1$, the solute concentration has a layer distinct from that of the temperature. This is as expected since δ_{SN} appears whenever the solute conduction term in the solute equation becomes important. For example, when $\tau \ll 1$, as we approach the boundary $z = 0$ from the interior the thermal and the solute conduction terms become important successively. The latter is $O(\tau)$ hence $\delta_{SN} \ll \delta_{tN}$.

It is clear that a sufficiently small $1-K$ stabilizes the flow. As $K \to 1$, α_n, F, F_S and N approach zero. However, for sufficiently large τ ($\gg 1$), N and F_S decrease, while F and α_n remain large until $N \to 0$ and the boundary-layer structure breaks down. Of course, care must also be exercised here, for the nonlinear regime of our problem is valid only asymptotically away from its stability regime.

The multi-modal analysis has shown that the solute concentration affects the boundary-layer structure only in the last mode α_N for the range $\tau \gg 1$ or $\tau \ll 1$. However, the rest of the α_n modes ($n = 1, \ldots, N-1$) have a regular structure similar to that in the pure thermal convection problem (I). The main purpose of using the multi-modal analysis is to determine the proper optimal flow quantities on the basis of the mean-field equations for sufficiently large R. The well-known layering problem of double-diffusive convection (Turner 1974) is not considered here. The important problem of layering has been observed experimentally, but has not yet been solved theoretically to a sufficient degree to understand the problem. The conjecture that the layering is a higher-mode phenomenon seems reasonable. Our theoretical result that only in the α_N mode does the solute concentration affect the boundary-layer structure for the transport supports such a conjecture.

Lindberg (1971) considered the upper-bound problem of turbulent thermohaline convection for the diffusive regime. Using the upper-bound technique of Howard (1963; 1968, unpublished work referred to by Lindberg), maximizing F subject to the so-called power integrals and considering only a single mode, he obtained the relationships

$$F = 0{\cdot}14R^{\frac{3}{8}}(1-R_S\tau^{\frac{5}{11}}/R)^{\frac{3}{8}}, \quad F_S = F\tau^{\frac{5}{11}}.$$

The relation for F is consistent with the earlier work on thermal convection for

S_y	$\log_{10}R$	F (present)	F (Lindberg)	F_S (present)	F_S (Lindberg)
0·01	5	7·54	10·08	0·38	1·24
	6	15·60	23·91	0·78	2·95
	7	32·10	56·70	1·61	6·99
	8	65·78	134·46	3·30	16·58
	9	134·38	318·84	6·75	39·31
	10	273·85	756·10	13·76	93·22
0·03	5	7·21	8·62	0·36	1·06
	6	14·93	20·44	0·75	2·52
	7	30·73	48·48	1·54	5·98
	8	62·99	114·96	3·16	14·17
	9	128·71	272·63	6·45	33·61
	10	262·32	646·51	13·14	79·70
0·07	5	6·72	5·55	0·34	0·68
	6	13·92	13·16	0·69	1·62
	7	28·68	31·21	1·43	3·85
	8	58·80	74·01	2·93	9·12
	9	120·18	175·52	5·99	21·64
	10	244·99	416·21	12·22	51·31

TABLE 1. Comparison of values of F and F_S from the present study and Lindberg (1971) for τ = 0·01 and $S_y = R_S\tau/R$ = 0·01, 0·03 and 0·07.

$R_S\tau^{\frac{5}{11}}/R \ll 1$ (Howard 1963, 1968). Lindberg plotted F *vs.* R (on a logarithmic scale) in his figure 1 for $\tau = 0{\cdot}01$ and for different values of the stability number $S_y = R_S\tau^{\frac{5}{11}}/R$. Since our mean-field equations contain the power integrals used by Lindberg, we expect that our upper-bound results are closer to the true upper bounds. By true upper bounds we mean the upper bounds on the flow quantities derived by maximizing F subject to the full equations of motion, heat and solute. Of course this expectation should hold for sufficiently large R and Pr. Table 1 gives a direct comparison between the optimal values F and F_S obtained by Lindberg and in present study (for $\tau = 0{\cdot}01$). We have considered the integer values $\log_{10}R = 5, \ldots, 10$ for the three different values 0·01, 0·03 and 0·07 of the stability number $S_y = R_S\tau/R$. It can be easily seen that the single-mode solution of the present study gives better upper bounds throughout the ranges of R and S_y considered in table 1.

The following conclusions, which are all as expected, can be made from table 1.

(i) The values of F and F_S from the present study are in general smaller than the corresponding values from Lindberg's study.

(ii) Lindberg's stability number is larger than that in our study. Thus for given τ and R, as R_S increases Lindberg's F and F_S approach zero sooner than those from our study. This result should not be misinterpreted. The present study and that of Lindberg are supposed to be valid asymptotically. Therefore, for slow convective motion, we should not expect to obtain quantitative agreement with what actually happens.

(iii) F_S is considerably smaller than F.

(iv) F and F_S increase with R for a given stability number. For $(1-K)R \sim 10^{10}$, it can easily be shown that the optimal nonlinear convection is generally characterized by a double-mode solution.

S_y	$\log_{10}R^*$	F_S (present)	F_S (Straus)	F (present)
0·01	5	7·36	9·04	0·74
	6	15·23	21·43	1·52
	7	31·34	50·83	3·13
	8	64·24	120·53	6·42
	9	131·25	285·82	13·12
	10	267·48	677·78	26·75
0·13	5	5·37	6·48	0·21
	6	11·14	15·36	0·45
	7	22·97	36·43	0·92
	8	47·12	86·40	1·88
	9	96·33	204·88	3·85
	10	196·44	485·84	7·86
0·63	5	1·19	1·28	0·02
	6	2·47	3·04	0·03
	7	5·12	7·22	0·07
	8	10·53	17·12	0·15
	9	21·58	40·60	0·31
	10	44·08	96·27	0·63

TABLE 2. Comparison of values of F_S from the present study and Straus (1974) for $\tau = 0{\cdot}01$ and $S_y = R/R^* = 0{\cdot}01$, $0{\cdot}13$ and $0{\cdot}63$.

Straus (1972) considered the problem of finite amplitude double-diffusive convection in the salt-finger regime. Free boundaries, $Pr \gg 1$, $\tau \ll 1$ and constant mean gradients of temperature and solute were assumed. Thus for sufficiently small τ and large Pr, the nonlinearities in the momentum and heat equations could be ignored. The important transport quantity was the solute. The motion was two-dimensional, and moderate values of R and $R^* = R_S/\tau$ were assumed. He showed, in particular, that the wavenumber for which F_S was maximized increased considerably in the double-diffusive case ($R \neq 0$) as R^* increased away from its critical stability value R_C^*. This wavenumber was found to be an increasing function of R for a given R^*. For $R \geqslant 10^4$, the solute flux was approximated by

$$F_S = 0{\cdot}11 R^{*0{\cdot}36}(1 - R_C^*/R^*)^{1{\cdot}36}.$$

In order to compare the present work with these results, we modify our model for the salt-finger regime by interchanging the role of temperature and solute. The equations in § 3 are applicable after replacing F, F_S, R, R_S, τ, δ_{tN} and δ_{SN} by F_S, F, R_S/τ, R/τ, $1/\tau$, δ_{SN} and δ_{tN} respectively. Using (32) and (34) for $n = 1$ (which is appropriate for the comparison with Straus' results) and $\tau \ll 1$, we have

$$\alpha_1^4 = \gamma(1-K)R^*, \quad F_S = (\tfrac{1}{400}\gamma)^{0{\cdot}1}[(1-\gamma)/2{\cdot}2212]^{1{\cdot}2}[(1-K)R^*]^{0{\cdot}3}[\log((1-K)R^*)]^{0{\cdot}2},$$

where

$$K = R/(\gamma R^*), \quad \gamma = \tfrac{1}{26}\{(1+10R/R^*) + [(1+10R/R^*)^2 + 104R/R^*]^{0{\cdot}5}\}.$$

It can be shown that α_1 increases with R for a given R^* and also increases with R^* for a given R. For $R^* \gg R$, the solute flux satisfies

$$F_S \propto R^{*0{\cdot}33}(\log R^*)^{0{\cdot}2}.$$

This functional dependence of F_S on R^* is close to that of Straus for $R^* \gg R^*_C$. Our study is based on the maximized nonlinear asymptotic state ($R \to \infty$), whereas Straus' study is concerned with small but finite amplitudes for moderate values of R. It is not expected that there will be many other similarities between the results of these two studies.

In a more recent study of the salt-finger regime, Straus (1974) considered the upper-bound problem by using Howard's (1963) technique. He maximized the solute flux subject to the power integrals derived from the solute and momentum equations and the linearized heat equation under the assumption that τ was sufficiently small. He also considered a single-mode solution and obtained

$$F_S = fR^{*-1}R^{\frac{11}{8}},$$

where f is a function of R^*/R only. As for our comparison with Lindberg's study, discussed above, we expect that our upper-bound results (modified for the salt-finger regime) are closer to the true upper bounds than those of Straus. Table 2 gives a direct comparison between the optimal values of F_S obtained here and those of Straus (for $\tau = 0{\cdot}01$). These results indicate that F may become significant at larger R_S, and hence the assumption used by Straus to ignore F may no longer hold. We consider here integer values $\log_{10} R^* = 5, \ldots, 10$ for the three different values 0·01, 0·13 and 0·63 of the stability number $S_y = R/R^*$. It can be easily seen that our single-mode solution gives better upper bounds throughout the ranges of R^* and S_y considered in table 2.

The following conclusions, which are all as expected, can be made from table 2.

(i) Our values of F_S are considerably smaller than the corresponding values from Straus' study.

(ii) F is considerably smaller than F_S.

(iii) F_S and F increase with R^* for a given S_y.

The author would like to thank Professor C. Y. Chan and the referees for their comments. This work was partially supported by the Office of Naval Research, Contract No. NOOO14–67–A–0235–008.

REFERENCES

Busse, F. H. 1969 *J. Fluid Mech.* **37**, 457.
Busse, F. H. 1970 *Max-Planck-Inst. Phys. Astrophys. Rep.* MPI-PAE/Astro 31.
Busse, F. H. & Joseph, D. D. 1972 *J. Fluid Mech.* **54**, 521.
Chan, S. K. 1971 *Stud. Appl. Math.* **50**, 13.
Chan, S. K. 1974 *J. Fluid Mech.* **64**, 477.
Degens, E. T. & Ross, D. A. (eds.) 1969 *Hot Brines and Recent Heavy Metal Deposits in the Red Sea.* Springer.
Gupta, V. P. & Joseph, D. D. 1973 *J. Fluid Mech.* **57**, 491.
Herring, J. R. 1963 *J. Atmos. Sci.* **20**, 325.
Howard, L. N. 1963 *J. Fluid Mech.* **17**, 405.
Lindberg, W. R. 1971 *J. Phys. Ocean.* **1**, 187.
Malkus, W. V. R. 1954 *Proc. Roy. Soc.* A **225**, 196.
Neal, V. T., Neshyba, S. & Denner, W. 1969 *Science* **166**, 373.
Neshyba, S., Neal, V. T. & Denner, W. 1971 *J. Geophys. Res.* **76**, 8117.
Riahi, N. 1977 *J. Fluid Mech.* **81**, 523.

STRAUS, J. 1972 *J. Fluid Mech.* **56**, 353.
STRAUS, J. 1974 *Phys. Fluids* **17**, 520.
TURNER, J. S. 1973 *Buoyancy Effects in Fluids*. Cambridge University Press.
TURNER, J. S. 1974 *Ann. Rev. Fluid Mech.* **6**, 37.
VERONIS, G. 1965 *J. Mar. Res.* **23**, 1.
VERONIS, G. 1968 *J. Fluid Mech.* **34**, 315.